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Dynamic Stiffness Approach to Frequency Analysis of FGM Beam Bonded with a Piezoelectric Layer

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Abstract
Frequency analysis of a functionally graded beam bonded by a piezoelectric layer is carried out by using the dynamic stiffness method. First, governing equations are conducted for FGM beam element with a piezoelectric layer based on the Timoshenko beam theory and power law of material grading. The established equations are solved to get frequency dependent shape functions that are employed then for constructing dynamic stiffness matrix of the beam element. Natural frequencies of an FGM beam with a piezoelectric layer representing a distributed sensor are examined in dependence upon the material properties and thickness of the piezoelectric layer

Keywords: *FGM Beam, Piezoelectric Material, Dynamic Stiffness Method.*

1. Introduction

Piezoelectric material has got more and more used for control of engineering structures [1-3] due to the property that couples its elastic evolution with electric field. Particularly, it is a specially prevailing device for structural health monitoring as both sensor and actuators [4-6].

The authors of References [7,8] have shown that piezoelectric patch can be efficiently used also for repairing a notched or cracked beam. Since a typical property of the piezoelectric material is its undergoing at high frequency band, the frequency domain methods are most appropriate in modelling piezoelectric structures. Lee and his co-workers [9,10] and Park et al [11] have extended the spectral element method for composite beam with piezoelectric layers. Li and Shi studied free vibration of a functionally graded piezoelectric beam via state-space based differential quadrature [12].

In this report, the dynamic stiffness method is developed for frequency analysis of functionally graded material (FGM) beam with piezoelectric layer. This is first effort of the authors to apply the piezoelectric material for health monitoring FGM structures.

2. Governing equations

2.1. Constitutive equations for base FGM beam

Suppose that FGM properties of the base beam are varying accordingly to the power law

$$\{E(z), G(z), \rho(z)\} = \{E_b, G_b, \rho_b\} + +\{E_t - E_b, G_t - G_b, \rho_t - \rho_b\}(z/h + 1/2)^n \qquad (2.1)$$

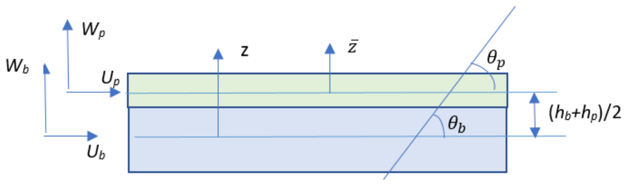

Fig.1. FGM beam with piezoelectric layer

where *E*, *G*, ρ denote Young's, shear modulus and mass density of the material respectively, the subscripts *b* and *t* indicate bottom and top material components and *z* is the ordinate measured from the central axis of the beam. Following Timoshenko beam theory, the constitutive equations are

$$u_b(x,z,t) = u_{b0}(x,t) - z\theta_b(x,t);$$
$$w_b(x,z,t) = w_{b0}(x,t); \qquad (2.2)$$
$$\varepsilon_{bx} = u'_{b0} - z\theta'_b; \gamma_b = w'_{b0} - \theta_b;$$
$$\sigma_{bx} = E(z)\varepsilon_{px}; \tau_b = \kappa G(z)\gamma_b, \qquad (2.3)$$

where the index *b* at components in the deformable field now denote those of the base beam, the index 0 implies that is measured at the mid-plan of the beam. Using the equations strain energy of the base beam can be calculated as

$$\Pi_b = \iiint (\sigma_{bx}\varepsilon_{bx} + \tau_b\gamma_b)dV_b = \iiint [E(z)\varepsilon_{bx}^2 + \kappa G(z)\gamma_b^2]dV_b = \int_0^L \left\{ \begin{array}{c} A_{11}u'^2_{b0} - 2A_{12}u'_{b0}\theta'_b + A_{22}\theta'^2_b + \\ +A_{33}(w'_{b0} - \theta_b)^2 \end{array} \right\} dx \qquad (2.4)$$

where

$$(A_{11}, A_{12}, A_{22}) = b\int_{-h_b/2}^{h_b/2} E(z)(1, z, z^2)dz;$$
$$A_{33} = b\kappa \int_{-h_b/2}^{h_b/2} G(z)dz \qquad (2.5)$$

2.2. Constitutive equations for piezo-electric layer

Let's consider the piezoelectric layer as a Timoshenko beam element, so that constitutive equations can be expressed as

$$u_p(x,\bar{z},t) = u_{p0}(x,t) - \bar{z}\theta_p(x,t),$$
$$w_p(x,\bar{z},t) = w_{p0}(x,t)$$
$$\varepsilon_{px} = u'_{p0} - \bar{z}\theta'_p, \gamma_p = w'_{p0} - \theta_p$$

The piezoelectric material is described by

$$\sigma_{px} = C^p_{11}\varepsilon_{px} - h_{13}D; \tau_p = C^p_{55}\gamma_p;$$
$$\in = -h_{13}\varepsilon_{px} + \beta^p_{33}D \quad (2.6)$$

where C^p_{11}, h_{13}, β^p_{33} are elastic modulus, piezoelectric and dielectric constants respectively. $\in$ and D are electric field and displacement of the piezoelectric layer.

Perfect bonding between the base beam and piezo-electric layer is represented by the conditions

$$u_b\left(x,\frac{h_b}{2},t\right) = u_p\left(x,-\frac{h_p}{2},t\right),$$
$$w_b(x,h_b/2,) = w_p(x,-h_p/2,t) \quad (2.7)$$

that yield

$$u_{p0} = u_{b0} - \theta_b h/2, h = h_b + h_p,$$
$$w_{p0} = w_{b0}, \theta_b = \theta_p \quad (2.8)$$

Therefore

$$\varepsilon_{px} = u'_{b0} - (\bar{z} + h/2)\theta'_b, \gamma_p = w'_{b0} - \theta_b; \quad (2.9)$$

and

$$\Pi_p = (1/2)\iiint(\sigma_{px}\varepsilon_{px} + \tau_p\gamma_p + \in D)dV_p$$
$$= (1/2)\iiint[C^p_{11}\varepsilon^2_{px} - 2h_{13}D\varepsilon_{px} + C^p_{55}\gamma^2_p + \beta^p_{33}D^2]dV_p$$
$$= \frac{1}{2}\int_0^L \left\{ \begin{matrix} C^p_{11}A_p u'^2_{b0} - 2C^p_{11}A_p h u'_{b0}\theta'_b + \\ C^p_{11}(I_p + A_p h^2/4)\theta'^2_b + C^p_{55}A_p(w'_{b0} - \theta_b)^2 \end{matrix} \right\} dx$$
$$-(1/2)\int_0^L \{2h_{13}A_p D(u'_{b0} - h\theta'_b/2) - \beta^p_{33}A_p D^2\}dx. \quad (2.10)$$

2.3. Equations of motion for FGM beam element with a piezoelectric layer

Using Hamilton's principle, equations of motion of the electromechanical system are established as

$$A^*_{11}u''_{b0} - I^*_{11}\ddot{u}_{b0} - A^*_{12}\theta''_b + I^*_{12}\ddot{\theta}_b = 0;$$
$$-A^*_{12}u''_{b0} + I^*_{12}\ddot{u}_{b0} + A^*_{22}\theta''_b - I^*_{22}\ddot{\theta}_b +$$
$$+A^*_{33}(w'_{b0} - \theta_b) = 0;$$
$$A^*_{33}(w''_{b0} - \theta'_b) - I^*_{11}\ddot{w}_{b0} = 0;$$
$$D = [h_{13}/\beta^p_{33}](u'_{b0} - h\theta'_b/2) \quad (2.11)$$

where the following notations have been used

$$A^*_{11} = A_{11} + E_pA_p; \; I^*_{11} = I_{11} + \rho_pA_p;$$
$$A^*_{12} = A_{12} + E_pA_ph/2; \; I^*_{12} = I_{12} + \rho_pA_ph/2;$$
$$A^*_{22} = A_{22} + E_pI_p + E_pA_ph^2/4; \quad (2.12)$$
$$I^*_{22} = I_{22} + \rho_pI_p + \rho_pA_ph^2/4$$
$$A^*_{33} = \kappa A_{33} + C^p_{55}A_p; \; A_p = bh_p; \; I_p = bh^3_p/12;$$
$$E_p = C^p_{11} - h^2_{13}/\beta^p_{33}; (I_{11}, I_{12}, I_{22}) =$$
$$= b\int_{-h_b/2}^{h_b/2}\rho(z)(1,z,z^2)dz. \quad (2.14)$$

The resultant forces are determined for the beam as

$$N = A^*_{11}u'_{b0} - A^*_{12}\theta'_b; \; Q = A^*_{33}(w'_{b0} - \theta_b);$$
$$M = A^*_{12}u'_{b0} - A^*_{22}\theta'_b. \quad (2.15)$$

Obviously, in case of beam without piezoelectric layer all the constants $A^*_{11}, A^*_{12}, A^*_{22}, A^*_{33}, I^*_{11}, I^*_{12}, I^*_{22}$ become those of the host FGM beam A_{11}, A_{12}, A_{22}, A_{33}, I_{11}, I_{12}, I_{22} (without asterisk superscript).

3. Dynamic stiffness model

3.1. General vibration mode shape

Seeking solution of Eq. (2.11) in the form

$$u_{b0}(x,t) = U_0e^{\lambda x + i\omega t};$$
$$\theta_b(x,t) = \Theta_0e^{\lambda x + i\omega t};$$
$$w_{b0}(x,t) = W_0e^{\lambda x + i\omega t}; \quad (3.1)$$

leads that equation to

$$\begin{bmatrix} A^*_{11}\lambda^2 + \omega^2I^*_{11} & -A^*_{12}\lambda^2 - \omega^2I^*_{12} & 0 \\ -A^*_{12}\lambda^2 - \omega^2I^*_{12} & A^*_{22}\lambda^2 + \omega^2I^*_{22} - A^*_{33} & A^*_{33}\lambda \\ 0 & -A^*_{33}\lambda & A^*_{33}\lambda^2 + \omega^2I^*_{11} \end{bmatrix} \begin{Bmatrix} U_0 \\ \Theta_0 \\ W_0 \end{Bmatrix} = 0 \quad (3.2)$$

So that one obtains characteristic equation for seeking the wave number λ as follows

$$det\begin{bmatrix} A^*_{11}\lambda^2 + \omega^2I^*_{11} & -A^*_{12}\lambda^2 - \omega^2I^*_{12} & 0 \\ -A^*_{12}\lambda^2 - \omega^2I^*_{12} & A^*_{22}\lambda^2 + \omega^2I^*_{22} - A^*_{33} & A^*_{33}\lambda \\ 0 & -A^*_{33}\lambda & A^*_{33}\lambda^2 + \omega^2I^*_{11} \end{bmatrix} = 0$$

or

$$a\lambda^6 + b\lambda^4 + c\lambda^2 + d = 0. \quad (3.3)$$

Let roots of Eq. (3.3) with respect to λ be found in the form

$$\lambda_1 = k_1 = \sqrt{\eta_1}, \lambda_2 = k_2 = \sqrt{\eta_2}, \lambda_3 = k_3 = \sqrt{\eta_3},$$
$$\lambda_4 = -k_1, \lambda_5 = -k_2, \lambda_6 = -k_3, \quad (3.4)$$

where η_1, η_2, η_3 are roots of the cubic equation $a\eta^3 + b\eta^2 + c\eta + d = 0$. Hence, so-called vibration mode shape $U(x,\omega) = U_0e^{\lambda x}, \Theta(x,\omega) = \Theta_0e^{\lambda x}, W(x,\omega) = W_0e^{\lambda x}$ can be expressed as

$$U(x,\omega) = C_1e^{k_1x} + C_2e^{k_2x} + C_3e^{k_3x} + C_4e^{-k_1x} +$$
$$+C_5e^{-k_2x} + C_6e^{-k_3x};$$
$$\Theta(x,\omega) = \alpha_1C_1e^{k_1x} + \alpha_2C_2e^{k_2x} + \alpha_3C_3e^{k_3x} +$$
$$+\alpha_1C_4e^{-k_1x} + \alpha_2C_5e^{-k_2x} + \alpha_3C_6e^{-k_3x};$$
$$W(x,\omega) = \beta_1C_1e^{k_1x} + \beta_2C_2e^{k_2x} + \beta_3C_3e^{k_3x} -$$
$$-\beta_1C_4e^{-k_1x} - \beta_2C_5e^{-k_2x} - \beta_3C_6e^{-k_3x}. \quad (3.5)$$

where

$$\alpha_j = \frac{\omega^2I^*_{11} + \eta_jA^*_{11}}{\omega^2I^*_{12} + \eta_jA^*_{12}}; \beta_j = \frac{(\omega^2I^*_{11} + \eta_jA^*_{11})\lambda_jA^*_{33}}{(\omega^2I^*_{12} + \eta_jA^*_{12})(\omega^2I^*_{11} + \eta_jA^*_{33})};$$
$$j = 1,2,,3. \quad (3.6)$$

3.2. Dynamic stiffness matrix

Using the shape functions (3.5), vector of nodal displacements $\mathbf{U}_e = \{U_1(\omega), \ldots, U_6(\omega)\}$, defined in Fig. 2, can be calculated as

$$\{\mathbf{U}_e\} = [\mathbf{\Phi}]\{\mathbf{C}\}, \quad (3.7)$$

Where $\mathbf{C} = \{C_1, \ldots, C_6\}$ and

$$[\Phi]=\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 & -\beta_1 & -\beta_2 & -\beta_3 \\ \alpha_1 e^{k_1L} & \alpha_2 e^{k_2L} & \alpha_3 e^{k_3L} & \alpha_1 e^{-k_1L} & e^{-k_2L} & e^{-k_3L} \\ e^{k_1L} & e^{k_2L} & e^{k_3L} & e^{-k_1L} & e^{-k_2L} & e^{-k_3L} \\ \beta_1 e^{k_1L} & \beta_2 e^{k_2L} & \beta_3 e^{k_3L} & -\beta_1 e^{-k_1L} & -\beta_2 e^{-k_2L} & -\beta_3 e^{-k_3L} \end{bmatrix} \quad (3.9)$$

$$U_1 = U(0,\omega),\ U_2 = \Theta(0,\omega),\ U_3 = W(0,\omega)$$
$$U_4 = U(L,\omega),\ U_5 = \Theta(L,\omega),\ U_6 = W(L,\omega) \quad (3.8)$$

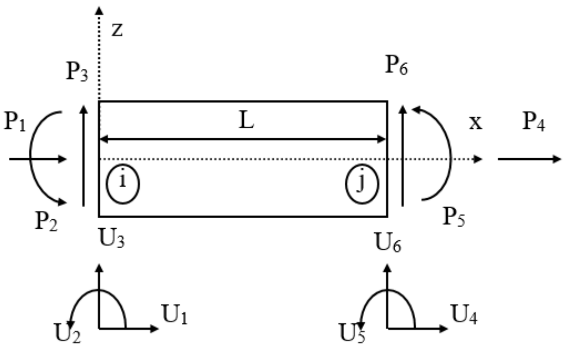

Fig. 2. Nodal displacements and forces of a beam element

Similarly, the internal forces

$$N(x,\omega) = A_{11}^{*}U'(x,\omega) - A_{12}^{*}\Theta'(x,\omega);$$
$$Q(x,\omega) = A_{33}^{*}[W'(x,\omega) - \Theta(x,\omega)];$$
$$M(x,\omega) = A_{12}^{*}U'(x,\omega) - A_{22}^{*}\Theta'(x,\omega) \quad (3.10)$$

can be calculated at the nodes as

$$\{\mathbf{P}_e\} = [\mathbf{\Psi}]\{\mathbf{C}\} \quad (3.11)$$

where $\mathbf{P}_e = \{P_1(\omega), \dots, P_6(\omega)\}$ and

$$P_1 = N(0,\omega), P_2 = M(0,\omega), P_3 = Q(0,\omega),$$
$$P_4 = N(L,\omega), P_5 = M(L,\omega), P_6 = Q(L,\omega). \quad (3.12)$$

The matrix $[\mathbf{\Psi}]$ in Eq. (3.11) has the elements

$$\psi_{11} = A_{11}^{*}k_1\alpha_1 - A_{12}^{*}k_1;\ \psi_{12} = A_{11}^{*}k_2\alpha_2 - A_{12}^{*}k_2;$$
$$\psi_{13} = A_{11}^{*}k_3\alpha_3 - A_{12}^{*}k_3;\ \psi_{14} = -A_{11}^{*}k_1\alpha_1 + A_{12}^{*}k_1;$$
$$\psi_{15} = -A_{11}^{*}k_2\alpha_2 + A_{12}^{*}k_2; \psi_{16} = -A_{11}^{*}k_3\alpha_3 + A_{12}^{*}k_3$$
$$\psi_{21} = A_{12}^{*}k_1\alpha_1 - A_{22}^{*}k_1;\ \psi_{22} = A_{12}^{*}k_2\alpha_2 - A_{22}^{*}k_2;$$
$$\psi_{23} = A_{12}^{*}k_3\alpha_3 - A_{22}^{*}k_3;\ \psi_{24} = -A_{12}^{*}k_1\alpha_1 + A_{22}^{*}k_1;$$
$$\psi_{25} = -A_{12}^{*}k_2\alpha_2 + A_{22}^{*}k_2; \psi_{26} = -A_{12}^{*}k_3\alpha_3 + A_{22}^{*}k_3$$
$$\psi_{31} = A_{33}^{*}(\beta_1k_1\alpha_1 - 1);\ \psi_{32} = A_{33}^{*}(\beta_2k_2\alpha_2 - 1);$$
$$\psi_{33} = A_{33}^{*}(\beta_3k_3\alpha_3 - 1); \psi_{34} = A_{33}^{*}(\beta_1k_1\alpha_1 - 1);$$
$$\psi_{35} = A_{33}^{*}(\beta_2k_2\alpha_2 - 1); \psi_{36} = A_{33}^{*}(\beta_3k_3\alpha_3 - 1);$$
$$\psi_{41} = (A_{11}^{*}k_1\alpha_1 - A_{12}^{*}k_1)e^{k_1L};$$
$$\psi_{42} = (A_{11}^{*}k_2\alpha_2 - A_{12}^{*}k_2)e^{k_2L};$$
$$\psi_{43} = (A_{11}^{*}k_3\alpha_3 - A_{12}^{*}k_3)e^{k_3L};$$
$$\psi_{44} = (A_{12}^{*}k_1 - A_{11}^{*}\alpha_1k_1)e^{-k_1L};$$
$$\psi_{45} = (A_{12}^{*}k_2 - A_{11}^{*}\alpha_2k_2)e^{-k_2L};$$
$$\psi_{46} = (A_{12}^{*}k_3 - A_{11}^{*}\alpha_3k_3)e^{-k_3L};$$
$$\psi_{51} = (A_{12}^{*}\alpha_1k_1 - A_{22}^{*}k_1)e^{k_1L};$$
$$\psi_{52} = (A_{12}^{*}k_2\alpha_2 - A_{22}^{*}k_2)e^{k_2L};$$
$$\psi_{53} = (A_{12}^{*}k_3\alpha_3 - A_{22}^{*}k_3)e^{k_3L};$$
$$\psi_{54} = (A_{22}^{*}k_1 - A_{12}^{*}\alpha_1k_1)e^{-k_1L};$$
$$\psi_{55} = (A_{22}^{*}k_2 - A_{12}^{*}\alpha_2k_2)e^{-k_2L};$$
$$\psi_{56} = (A_{22}^{*}k_3 - A_{12}^{*}\alpha_3k_3)e^{-k_3L};$$
$$\psi_{61} = A_{33}^{*}(\beta_1k_1 - 1)e^{k_1L};$$
$$\psi_{62} = A_{33}^{*}(\beta_2k_2 - 1)e^{k_2L};$$
$$\psi_{63} = A_{33}^{*}(\beta_3k_3 - 1)e^{k_3L};$$
$$\psi_{64} = A_{33}^{*}(\beta_1k_1 - 1)e^{-k_1L};$$
$$\psi_{65} = A_{33}^{*}(\beta_2k_2 - 1)e^{-k_2L};$$
$$\psi_{66} = A_{33}^{*}(\beta_3k_3 - 1)e^{-k_3L}. \quad (3.13)$$

Eliminating constant vector C from Eqs. (3.7) and (3.11) gives

$$\{\mathbf{P}_e\} = [\mathbf{\Psi\Phi}^{-1}]\{\mathbf{U}_e\} = [\mathbf{D}_e(\boldsymbol{\omega})]\{\mathbf{U}_e\} \quad (3.14)$$

with matrix $\mathbf{D}(\boldsymbol{\omega})$ that is acknowledged as the dynamic stiffness matrix of the beam element.

For a structure composed off a number of such the beam elements, the total dynamic stiffness matrix is assembled from those of the component elements by

$$[\mathbf{D}(\omega)] = \textstyle\sum_e \mathbf{T}_e^{t}\mathbf{D}_e(\boldsymbol{\omega})\mathbf{T}_e \quad (3.15)$$

3.3. Modal analysis of FGM beam bonded by piezoelectric layer

Free vibration of a structure formulated by using the dynamic stiffness model is governed by solving the equation

$$[\mathbf{D}(\omega)]\{\mathbf{U}\} = 0 \quad (3.16)$$

where total vector of nodal displacements $\{\mathbf{U}\}$ is assembled also from the local vector $\{\mathbf{U}_e\}$. Solution of Eq. (3.16) gives natural frequencies $\omega_1 . \omega_2, \dots$ determined as positive roots of equation

$$det[\mathbf{D}(\omega)] = 0 \quad (3.17)$$

and normalized solution $\bar{\mathbf{U}}(j)$ of equation

$$[\mathbf{D}(\omega_j)]\{\mathbf{U}\} = 0 \quad (3.18)$$

Therefore, mode shape corresponding to natural frequency ω_j would be determined by

$$\begin{Bmatrix} U_j(x) \\ \Theta_j(x) \\ W_j(x) \end{Bmatrix} = \begin{bmatrix} \alpha_1e^{k_1x} & \alpha_2e^{k_2x} & \alpha_3e^{k_3x} & \alpha_1e^{-k_1x} & \alpha_2e^{-k_2x} & \alpha_3e^{-k_3x} \\ e^{k_1x} & e^{k_2x} & e^{k_3x} & e^{-k_1x} & e^{-k_2x} & e^{-k_3x} \\ \beta_1e^{k_1x} & \beta_2e^{k_2x} & \beta_3e^{k_3x} & -\beta_1e^{-k_1x} & -\beta_2e^{-k_2x} & -\beta_3e^{-k_3x} \end{bmatrix} \cdot [\Phi]^{-1}\{\bar{\mathbf{U}}_e(j)\}$$

4. Examples

Let's consider an FGM beam of length *L* bonded by a piezoelectric layer along all the beam length as shown in Fig. 3. Note that if the beam is clamped at the ends, for which boundary conditions are

$$U(0) = \Theta(0) = W(0) = U(L) = \Theta(L) = W(L) = 0$$

then Eq. (3.7) leads to

$$[\Phi]\{\mathbf{C}\} = 0 \quad (4.1)$$

The latter equation gives rise immediately the frequency equation for the beam

$$f(\omega) = det[\Phi]\{\mathbf{C}\} = 0 \quad (4.2)$$

Numerical computation is carried out for FGM with following parameters

E_1=390e9Pa; ρ_1=3960kg/m^3; μ_1=0.25;

E_2=210e9Pa; ρ_2=7800 kg/m^3; μ_2=0.31;

L=1.0m; b=0.1m; h=L/10

with varying volume distribution index *n*. The parameters of piezoelectric material are

H_{13}=-7.70394e8; ρ=7750; C_{11}=69.0084e9;

B_{33}=7.38857e7; C_{55}=21.0526e9.

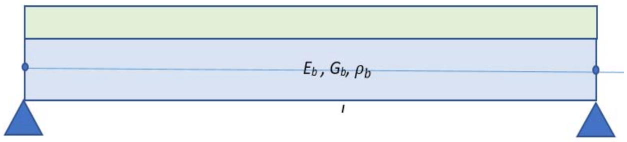

Fig. 3. FGM beam with a piezoelectric patch

Thickness of the piezoelectric layer varies from 0 (without piezoelectric layer) to 0.1 (the beam thickness) and boundary conditions for the beam are clamped ends. Results of the computation are depicted in Table 1 and Fig. 4.

Table 1. Variation of natural frequencies versus power law index *n* of FGM and thickness *h* of piezoelectric layer for clamped-clamped end beam

n	Freq.	hp = 0	hp = 0.01	hp = 0.02
	1	10.8205	10.2595	9.9140
	2	27.7924	26.3334	25.3985
0.1	3	50.3343	47.6680	45.8992
	4	56.3731	52.3676	49.1608
	5	76.5667	72.4797	69.6949
	1	10.2319	9.7532	9.4652
	2	26.2854	25.0404	24.2569
0.2	3	47.6118	45.3378	43.8498
	4	53.5455	50.0200	47.1606
	5	72.4358	68.9492	66.5975
	1	9.1182	8.7782	8.5895
	2	23.4254	22.5401	22.0174
0.5	3	42.4315	40.8149	39.8092
	4	48.0050	45.3109	43.0747
	5	64.5588	62.0764	60.4676
	1	8.2292	7.9864	7.8688
	2	21.1256	20.4903	20.1522
1.0	3	38.2389	37.0750	36.4073
	4	43.1884	41.1057	39.3475
	5	58.1469	56.3521	55.2607
	1	7.5376	7.3663	7.3015
	2	19.3119	18.8570	18.6522
2.0	3	34.8940	34.0499	33.6202
	4	38.9091	37.2893	35.9051
	5	52.9791	51.6631	50.9295
	1	6.9493	6.8451	6.8320
	2	17.7560	17.4632	17.3820
5.0	3	32.0050	31.4374	31.2166
	4	35.0282	33.7687	32.6832
	5	48.4886	47.5736	47.1418
	1	6.6339	6.5709	6.5922
	2	16.9432	16.7477	16.7462
10	3	30.5295	30.1234	30.0336
	4	33.3620	32.2413	31.2729
	5	46.2376	45.5513	45.3036

Table 1. Variation of natural frequencies versus power law index *n* of FGM and thickness h of piezoelectric layer for clamped-clamped end beam

n	Freq.	hp=0.05	hp=0.08	hp=0.1
	1	9.6862	10.0951	10.5041
	2	24.5118	25.0041	25.5356
0.1	3	42.4221	38.1190	36.0326
	4	43.8183	43.9338	44.2979
	5	65.9328	65.1768	65.0498
	1	9.3367	9.7948	10.2236
	2	23.6387	24.2734	24.8692
0.2	3	41.0480	37.0744	35.1281
	4	42.2748	42.6621	43.1463
	5	63.6249	63.2950	63.3530
	1	8.6362	9.1808	9.6429
	2	21.8717	22.7578	23.4656
0.5	3	38.1419	34.8241	33.1659
	4	39.1248	39.9972	40.6970
	5	58.8860	59.3243	59.7221
	1	8.0408	8.6423	9.1229
	2	20.3397	21.3910	22.1673
1.0	3	35.3747	32.6293	31.2342
	4	36.3465	37.5453	38.3877
	5	54.6515	55.6204	56.2574
	1	7.5627	8.1945	8.6783
	2	19.0653	20.2000	20.9988
2.0	3	32.7219	30.4779	29.3231
	4	33.9690	35.3399	36.2473
	5	50.9504	52.2180	52.9881
	1	7.1817	7.8352	8.3128
	2	17.9958	19.1716	19.9596
5.0	3	30.1578	28.3557	27.4216
	4	31.9003	33.3536	34.2684
	5	47.6508	49.0759	49.8996
	1	7.0083	7.6850	8.1632
	2	17.5053	18.7226	19.5086
10	3	29.0126	27.3954	26.5562
	4	30.9497	32.4704	33.3916
	5	46.1361	47.6676	48.5185

Observing the results given in Table 1 and graphs shown in Fig. 4 one can make the following notices:

- Natural frequencies of FGM beam bonded with a piezoelectric layer decrease with growing volume fraction index *n* independently upon thickness of the piezoelectric layer;
- Under growing thickness of piezoelectric layer from zero to thickness of the host beam, flexural vibration frequencies of the electro-mechanical system at first reduce to a minimum, then they get monotonically increasing. Curvature of the graphs at minimum value is increasing with reducing volume fraction index *n*.

- Longitudinal frequencies of the beam are all decreasing with growing thickness of piezoelectric layer regardless the index *n*.

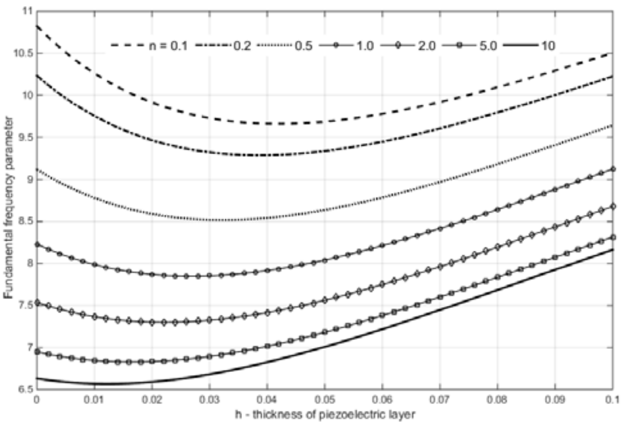

Fig. 4. Variation of fundamental frequency versus thickness of piezoelectric layer for clamped end FGM beam with n = 0.1, 0.2, 0.5, 1.0, 2.0, 5.0, 10.

The decrease and increase of natural frequency with growing thickness of piezoelectric layer can be explained as following: as well-known in the theory of vibration, natural frequency of a system is defined as ratio of its stiffness to mass. Therefore, frequency could be increasing or decreasing in dependence on whether the stiffness grow is more or less than that of the mass. Actually, as shown in Fig. 4, frequency of the beam bonded with piezoelectric layer is always less than that of beam without the bonded layer (when n = 0.1). This implies that contribution of the layer in stiffness is less than its contribution in mass. In the case when $n > 0.2$ the contribution in stiffness is less only for small thickness of bonded layer and it gets more than mass contribution from a value of the thickness that results in increasing natural frequency.

5. Conclusion

In the present report, governing equations of FGM beam bonded by a piezoelectric layer have been derived on the base of Timoshenko beam theory, power law of material grading. The established equations have the form of single FGM beam with those coefficients added by the terms representing contribution of the piezoelectric layer. These equations are then solved to get frequency-dependent shape functions needed to develop the dynamic stiffness method for modal analysis of the double beam.

The developed method has been employed to frequency analysis of an FGM beam with piezoelectric layer. The analysis shows that the piezoelectric layer does not modify the well-known properties of FGM beam even thickness of the layer reaches to thickness of the base beam. However, thickness of the piezoelectric layer makes a remarkable influence on natural frequencies of the bonded beam. Namely, thickness of piezoelectric layer increasing from zero to a level makes flexural frequencies decreased before they get monotonical increase, while the axial vibration frequencies are unaffected by increasing of the thickness.

Next study of the authors is to investigate influence of piezoelectric layer on natural frequencies of cracked FGM beam.

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